▪ Vectors Analysis

(1) Find ∇F where :

(i) $F = x^3 + x\,e^y + 3y$ (ii) $F = x\sin x + y^3 + 3y$

(iii) $F = x^3 e^y + y\cos z$ (iv) $F = y^3 \ln x + \tan xz$

(2)Find the directional derivative of the function $F = x^2 y^3 z^4$ at the point (1, 2, 1) in the direction of the vector $\overline{U} = 2\,i - j + 2k$

(3) Find $\nabla.\overline{U}$ and $\nabla x\overline{U}$ where :

(i) $\overline{U} = (x^3 y)i - (z\,e^y)j + (3 + \sin z)k$

(ii) $\overline{U} = (x^3 + y)i + (z\ \ln y)j + (xy + \cos z)k$

(iii) $\overline{U} = (xyz)i - (1 + e^{xy})j + (z\ \sin z)k$

(iv) $\overline{U} = (yz)i - (xz\,)j + (x^3 + y^2)k$

(4) Find $\nabla.\overline{U}$ at the point (1, 1, 1) where : $\overline{U} = (xyz)i + (x + y + z\,)j + \ln(xyz)k$

(5) Find $\nabla x\overline{U}$ at the point (2, – 1, –1) , $\overline{U} = (xyz)i - (x + y + z\,)j + \ln(xyz)k$

(6)If $\overline{U} = (xz)i - (xy)j + (yz)k$ and $\overline{V} = (y)i + (z^3)j + (x^2)k$

Find $\nabla.\left(\overline{U} + \overline{V}\right)$, $\nabla.\left(\overline{U} - \overline{V}\right)$, $\nabla.\left(\overline{U}x\overline{V}\right)$, $\nabla x\left(\overline{U} + \overline{V}\right)$, $\nabla\left(\overline{U}.\overline{V}\right)$, $\nabla.\left(\nabla x\overline{U}\right)$

(7)Prove that $\nabla.\left(\nabla x\overline{U}\right)$ for any vector $\overline{U}$.

(8)Write the vector $\overline{U} = (xy)i + (z)j + (yz)k$ in the cylindrical coordinates(r, θ, z) and spherical coordinates (r, θ, φ) and find the Jacobian of each transformation.

(9) Show that the new system corresponding to the transformation (r, θ, φ):

$$x = (2 - r\cos\theta)\cos\varphi\,,\ \ y = (2 - r\cos\theta)\sin\varphi\,,\ \ z = r\sin\theta$$

is orthogonal and the scale factors are 1, r and $2 - r\cos\theta$.

(10) Show that the new system corresponding to the transformation (α, β, γ):

$$x = \frac{1}{2}(\alpha^2 - \beta^2),\ \ y = \alpha\,\beta,\ \ z = \gamma$$

is orthogonal and find the scale factors where $-\infty < \alpha < \infty,\ 0 < \beta < \infty,\ -\infty < \gamma < \infty$

(11)If $\overline{U}$ is a vector in the cylindrical coordinates (r, θ, z). Show that :

$$\nabla.\overline{U} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,u_1) + \frac{\partial u_2}{\partial \theta} + r\,\frac{\partial u_3}{\partial z}\right]$$

▪ Special Functions

Find the following integrals:

(1) $\int_0^\infty e^{-\sqrt{x}}\,dx$ (2) $\int_0^\infty e^{-x^4}\,dx$ (3) $\int_0^1 x^2\,(-\ln x)^4\,dx$

(4) $\int_0^\infty x^5 e^{-2x^2}\,dx$ (5) $\int_2^\infty e^{4x-x^2}\,dx$ (6) $\int_3^\infty e^{-x^2+6x}\,dx$

(7) $\int_0^1 \sqrt[3]{1-x^3}\,dx$ (8) $\int_0^3 x\,(27-x^3)^{\frac{1}{3}}\,dx$ (9) $\int_0^{\frac{\pi}{2}} \sec x\,.\sqrt{\tan x}\,dx$

(10) $\int_3^4 (x-3)^4\sqrt{4-y}\,dy$ (11) $\int_0^2 z\,(16-z^4)^{\frac{1}{4}}\,dz$ (12) $\int_0^{\frac{\pi}{2}} \sin t\,.\sqrt{\tan t}\,dt$

▪ Multiple Integrals

(1)Find the following integrals :

(i) $\int_0^1\int_0^2 (12x\,y)\,dx\,dy$ (ii) $\int_0^1\int_0^2 (12\,x^3 y)\,dy\,dx$ (iii) $\int_0^1\int_0^x (x+e^y)\,dy\,dx$

(iv) $\int_0^1\int_0^{2y} (2x\,y)\,dx\,dy$ (v) $\int_1^2\int_0^x (\,x^3-y)\,dy\,dx$ (vi) $\int_1^2\int_1^x \frac{x}{y}\,dy\,dx$

(2)Find the following integrals:

(i) $\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ (ii) $\int_{-1}^1\int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ (iii) $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

(iv) $\int_0^1\int_0^{\sqrt{1-y^2}} \frac{dx\,dy}{\sqrt{x^2+y^2}}$ (v) $\int_0^1\int_0^{\sqrt{1-y^2}} \frac{x\,y}{\sqrt{x^2+y^2}}\,dx\,dy$ (vi) $\int_0^2\int_0^{\sqrt{4-x^2}} \frac{x+y}{\sqrt{x^2+y^2}}\,dy\,dx$

(3)Find the following integrals:

(i) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ (ii) $\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}\,dx\,dy$ (iii) $\iint_D \frac{\sin(x^2+y^2)}{x^2+y^2}\,dx\,dy$

Where D is the region between the two circles : $x^2+y^2=1,\ \ x^2+y^2=4.$

(4)Find the following integrals :

(i) $\int_0^1\int_0^2\int_0^3 (xyz)\,dz\,dy\,dx$ (ii) $\int_0^1\int_0^2\int_0^3 (4x+2yz)\,dx\,dy\,dz$

(iii) $\int_0^1\int_0^x\int_0^{xy} (x^2\,y)\,dz\,dy\,dx$ (iv) $\int_0^1\int_0^{2x}\int_0^{x+y} (x+2z)\,dz\,dy\,dx$

(5) Find the integral : $\int_{(0,0)}^{(1,1)}(x+y)dx+(2x-3y)dy$ along the curves :

(a) y = x (b) $x^2=y$ (c) $x=y^2$ (d) $y=x^3$

(6) Find the integral : $\int_{(0,0)}^{(1,1)}(x+2y)dx+(2x-y)dy$ along the curves :

(a) y = x (b) $x^2=y$ (c) $x=y^2$ (d) $y=x^3$

(7) Find the integral : $\oint_C (xy)\,dx+(x+y)dy$ along the curve C given by :

(a) $x^2=y\ ,\ x=y^2$ (b) $x^2+y^2=4$

(8) Find the integral : $\oint_C (x+2y)\,dx+(x^2-y)dy$ where the curve C is the sides of the triangle of vertices : (0, 0), (2, 0), (2, 2).

(9) Find the integral : $\oint_C (2xy)\,dx+(x^2+y^3)dy$ where the curve C given by :

(a) $y=x\ ,\ \ y=x^2$ (b) $x^2+y^2=5$

(10) Find the integral : $\oint_C (3x^2y)\,dx+(x^3-2y)dy$ where the curve C is the sides of the rectangle of vertices : (0, 0), (2, 0), (2, 2), (0. 2).

(11)Verify Green's theorem for : $\oint_C (x+xy)\,dx+(xy)dy$

where C is: (i)the circle $x^2+y^2=1$

(ii)the ellipse $4x^2+9y^2=36$

(iii)the sides of the triangle of vertices (0, 0), (3, 0),(3, 3).

(12)Find the flux of the vector : $\overline{\mathrm{U}}=(\mathrm{xz})\mathrm{i}+(\mathrm{xy})\mathrm{j}+(\mathrm{yz})\mathrm{k}$ through the surface of the paraboloid $x^2+y^2+z=1,\ \ z\geq 0$.

(13)Verify the Gauss's theorem for the vector : $\overline{\mathrm{U}}=(2\mathrm{x}+\mathrm{z})\mathrm{i}+(\mathrm{x}+\mathrm{y})\mathrm{j}+(\mathrm{xy})\mathrm{k}$ through the surface of the paraboloid $x^2+y^2+z=4,\ \ z\geq 0$.

(14)Verify the Stoke's theorem for the vector : $\overline{\mathrm{U}}=(2\mathrm{xyz}-3\mathrm{y})\mathrm{i}+(\mathrm{x}^2\mathrm{z})\mathrm{j}+(\mathrm{x}^2\mathrm{y})\mathrm{k}$ through the surface of the semi-sphere $x^2+y^2+z^2=1,\ \ z\geq 0$

▪ Complex Analysis

(1)Determine and sketch the image of each region under the function $f(z) = \sin z$:

(i) $0 \le x \le 2\pi, \ 1 \le y \le 2$ (ii) $0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le 2$

(iii) $0 \le x \le \pi, \ 1 \le y \le 2$ (iv) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \ 0 \le y \le 2$

(2)Find and sketch the image of the region : $0 \le x \le \pi, \ 0 \le y \le 1$ under the function $f(z) = \cos z$.

(3)Find the image of the following regions under the function $f(z) = e^z$:

(i) $0 \le x \le 1, \ 0 \le y \le \frac{\pi}{2}$ (ii) $0 \le x \le \ln 3, \ 0 \le y \le \pi$

(4)Show that :

(i) $1 + \cos\theta + \cos 2\theta + \cdots = \frac{1}{2}$

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta \ldots = \frac{\sin\theta}{2(1-\cos\theta)}$

(5)Find the sum of the following series :

(i) $\sum_{n=1}^{m} \sin n\theta$ (ii) $\sum_{n=1}^{\infty} \cos n\theta$ (iii) $\sum_{n=1}^{\infty} (-1)^n \cos n\theta$ (iv) $\sum_{n=1}^{\infty} (-1)^n \sin n\theta$

(6)Determine which of the following functions are harmonic. For each harmonic function find its conjugate such that f(z) is analytic :

(i) $u = x\sin y - y\cos x$ (ii) $v = 3 + x^2 - y^2$

(iii) $u = x^2 + 2y - y^2$ (iv) $v = x^2 + 2x - y^2$

(7)Find u(x, y) , v(x, y) of each of the following functions and show that they satisfy Laplace equations :

(i) $f(z) = z + \sin 2z$ (ii) $f(z) = z^2 + 2\cosh 2z$

(iii) $f(z) = \ln 3 + \cos^2 z$ (iv) $f(z) = z + e^{2z}$

(8)Write the Maclurin's series of each of the following functions :

(i) $f(z) = \frac{z}{z^2-5z+6}$ (ii) $f(z) = \frac{z}{(z-1)(x+2)(z-3)}$

(iii) $f(z) = \frac{\ln(1+z)}{z}$ (iv) $f(z) = z\sin\frac{1}{z}$

(v) $f(z) = z + e^{\frac{2}{z}}$ (vi) $f(z) = z^3\cos\frac{1}{z}$

(9)Find the zeroes and their order of each of the following functions :

(i) $f(z) = z^4 + z^2$ (ii) $f(z) = z^4 - 16$ (iii) $f(z) = \frac{1}{z}\sin z^3$

(iv) $f(z) = e^{2z} - e^z$ (v) $f(z) = z(e^z - 1)$ (vi) $f(z) = z\cos z^2$

(10)Show that :

(i) $\underset{z=i}{\text{Res}}\ f(z) = \underset{z=-i}{\text{Res}}\ f(z) = \frac{1}{2}$ where $f(z) = \frac{z}{z^2+1}$

(ii) $\underset{z=\frac{\pi}{2}}{\text{Res}}\ f(z) = \underset{z=-\frac{\pi}{2}}{\text{Res}}\ f(z) = -1$ where $f(z) = \tan z$

(iii) $\underset{z=0}{\text{Res}}\ f(z) = \frac{1}{2}$ where $f(z) = \frac{1}{z+\sin z}$

(iv) $\underset{z=0}{\text{Res}}\ f(z) = 3$ where $f(z) = e^{\frac{3}{z}}$

(v) $\underset{z=0}{\text{Res}}\ f(z) = -\frac{1}{6}$ where $f(z) = z^2\sin\frac{1}{z}$

(vi) $\underset{z=0}{\text{Res}}\ f(z) = 1$ where $f(z) = \frac{\sin z}{z^2}$

(vii) $\underset{z=0}{\text{Res}}\ f(z) = 0$ where $f(z) = \frac{\sin z}{z^3}$

(11)Find the poles and their order , also residues, of each of the following functions :

(i) $f(z) = z + 3^z$ (ii) $f(z) = \ln z$ (iii) $f(z) = \cosh z$

(iv) $f(z) = \frac{e^z-1}{z}$ (v) $f(z) = \frac{\sin 2z}{z}$ (vi) $f(z) = \frac{z-\sin z}{z}$

(vii) $f(z) = \frac{e^z-1}{z^2}$ (viii) $f(z) = \frac{\sin z}{z^2-z}$ (ix) $f(z) = \tan 2z$

(x) $f(z) = \frac{e^z}{z^3}$ (xi) $f(z) = \frac{z+1}{(z-2)^4}$ (xii) $f(z) = \cot z$

(xiii) $f(z) = \frac{e^z-1}{z}$ (xiv) $f(z) = \frac{\tan z}{z}$ (xv) $f(z) = \frac{\ln(z+1)}{z}$

(12)If C is the ellipse : $z = 5\cos t + i\,4\sin t$. Show that :

(i) $\oint_C \frac{1}{z+9}dz = 0$ (ii) $\oint_C \frac{e^{2z}}{z-3\pi i}dz = 0$ (iii) $\oint_C \frac{\cosh 2z}{z+9i}dz = 0$

(iv) $\oint_C \frac{\ln(z-7)}{z^2+36}dz = 0$ (v) $\oint_C \frac{\cos z}{z^2-49}dz = 0$ (vi) $\oint_C \frac{\sinh 2z}{z-9i}dz = 0$

(13)If C is the circle : $|z| = 1$. Show that :

(i) $\oint_C \frac{1}{z}dz = 2\pi i$ (ii) $\oint_C \frac{1}{4z+i}dz = \frac{\pi}{2}i$ (iii) $\oint_C \frac{\cos z}{z}dz = 2\pi i$

(iv) $\oint_C \frac{e^z}{z^2} dz = 2\pi i$ (v) $\oint_C \frac{z^2}{(2z-5)} dz = 0$ (vi) $\oint_C \frac{4^z}{2z-1} dz = 4\pi i$

(14)If C is the circle : $|z| = 4$. Show that :

(i) $\oint_C \frac{z}{z^2-1} dz = 2\pi i$ (ii) $\oint_C \frac{z+1}{z^2(z+2)} dz = 0$

(iii) $\oint_C \frac{z^2}{(z^2+3z+2)^2} dz = 0$ (iv) $\oint_C \frac{1}{z(z-2)^3} dz = 0$

(v) $\oint_C \frac{1}{z^2+z+1} dz = 0$ (vi) $\oint_C \frac{1}{(z+1)^3} dz = 0$

(vii) $\oint_C \frac{z+2}{z(z+1)} dz = 2\pi\, i$ (viii) $\oint_C \frac{1}{z(z+1)(z+4)} dz = -\frac{\pi}{6}\, i$

(15)If C is the circle : $|z| = 1$. Find the integrals :

(i) $\oint_C \frac{z^3}{(2z-1)^2} dz$ (ii) $\oint_C \frac{\sin z}{4z-\pi} dz$ (iii) $\oint_C \frac{\cos z}{(4z+\pi)^2} dz$

(iv) $\oint_C \frac{e^z}{z^3(2z+1)} dz$ (v) $\oint_C \frac{\ln(z+5)}{z^2} dz$ (vi) $\oint_C \frac{\cosh z}{z^4} dz$

(16) Show that :

(i) $\int_0^{2\pi} \frac{1}{10-6\sin\theta} d\theta = \frac{\pi}{4}$ (ii) $\int_0^{2\pi} \frac{1}{3+\cos\theta+2\sin\theta} d\theta = \pi$

(iii) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{4}$ (iv) $\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \frac{5\pi}{32}$

(v) $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta = \frac{4\pi}{\sqrt{27}}$ (vi) $\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)} dx = \frac{\pi}{2} e^{-1+i}$

(vii) $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{\pi}{e^3}$ (viii) $\int_{-\infty}^{\infty} \frac{1}{(x+1)^2(x^2+9)} dx = \frac{\pi}{12}$

(ix) $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{3}$ (x) $\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+9)^2} dx = \frac{7\pi}{108\, e^6}$

(xi) $\int_0^{\infty} \frac{\ln x}{(x^2+1)^2} dx = -\frac{\pi}{4}$ (xii) $\int_{-\infty}^{\infty} \frac{(\ln x)^2}{1+x^2} dx = \frac{\pi^2}{8}$